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A Percolation Approach to the Kauffman Model

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A study is made of the spreading of damage in the random but deterministic Kauffman model on the square lattice with the spreading from one edge of the lattice. The critical value of the parameter p_c above which the system becomes chaotic is found to be $p_c \approx 0.298$. The possibility of suppression of the chaotic phase by noise is also studied. It is found that for $p \geqslant p_c$, an extremely large noise level $g > 0.99$ is required, if possible at all.

KEY WORDS: Cellular automata; Kauffman model; percolation.

1. INTRODUCTION

There has recently been much interest in the random Kauffman cellular automata problem.[(1–7)] In this model, each of N lattice sites contains a Boolean variable or spin σ which is either zero or unity and also a set of rules unique to that site. These rules, which are generated randomly, determine the state of the spin at any given site based on the spin states of neighboring sites. Therefore, on a square lattice, which is the situation considered here, each site has 2^4 or 16 rules. These rules are generated randomly at each site using a set of random numbers uniformly distributed in the interval 0 to 1 and comparing with a fixed parameter $p \leqslant 0.5$ which gives the probability that the spin at each site has value unity. For our square lattice, let $\sigma_{ij}^t = 0, 1$ denote the value of the spin at time t at site (i, j). Then the rule for the evolution of the Kauffman model is given by

$$\sigma_{ij}^{t+1} = f_{ij}(\sigma_{i,j-1}^t, \sigma_{i,j+1}^t, \sigma_{i-1,j}^t, \sigma_{i+1,j}^t; p) \qquad (1)$$

where f is a function that takes the value 0 or 1, and $p \leqslant 0.5$ (actually $p \leqslant 1$, but f is symmetric with respect to the value $p = 0.5$) gives the average

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fraction of times f takes the value 1 among the 16 possible configurations of its σ arguments. In the original Kauffman model, p is fixed at the value 0.5, but the number of input sites varies and these input sites are chosen randomly from among any of the N lattice sites. These rules, once determined for each site of the lattice, are fixed for all subsequent times. The evolution of the system is therefore deterministic.

In the analysis of the Kauffman model, the concept of "damage" plays an essential role. Suppose we iterate according to the same rules two configurations $\{\sigma_i\}$ and $\{\rho_i\}$ on the same square lattice G. Then at time t we denote a site i of another square lattice G' as "occupied" (1) if $\sigma_i(t)$ and $\rho_i(t)$ differ, whereas we call it "empty" (0) if they have the same value. The number of sites in G' with value unity denotes the actual damage $M_{\text{act}}(t)$ at time t. On another square lattice G'' we record the sites in which $\sigma_i(t)$ and $\rho_i(t)$ have differed at least once up to time t. That is, at time t a site in G'' has value 1 if the same site has value 1 at time $t-1$. Otherwise it has value 1 or 0, depending on whether $\sigma_i(t)$ and $\rho_i(t)$ differ or not at time t. The number of sites in G'' with value unity denotes the total damage $M_{\text{tot}}(t)$ at time t. As explained in Ref. 6, for $\{\sigma_i\}$ and $\{\rho_i\}$ differing initially by only one site, the actual damage occurs either all on even or all on odd sites on G' at all subsequent times. Therefore, occupied sites on G' form clusters only if next nearest neighbor sites are considered to belong to the cluster. This problem does not occur for the total damage on G'', so that we can consider clusters formed by nearest neighbor sites.

In most earlier simulations of the random Kauffman model,[4–6] the "seed" used for the initial damage was taken to be a single site. Recently Grassberger,[8] in simulating percolation clusters, found that there are advantage in taking the seed as a $(d-1)$-dimensional hyperplane in a d-dimensional lattice. The main advantage is that one has larger clusters from the very beginning, so that fluctuations are small. When starting from a single seed, one has very large fluctuations provided the clusters are still small. Also, if one starts from a hyperplane, with systems of finite sizes L, and obtains quantities such as the threshold $p_c(L)$ depending on L, one can then use finite-size scaling to extrapolate to the $L \to \infty$ limit to obtain $p_c(\infty)$. In this paper we study the random Kauffman model on square lattice using cells of size $L \times L$ with L up to 200. For these cells we apply periodic boundary conditions in one direction and helical boundary conditions in the other. For studying the spreading of damage, the seed used is a line of length L at one edge of the cell. In addition, we introduce noise into the model and study its critical behavior. We find that the chaotic behavior can be suppressed, if at all, only at extremely large noise levels.

2. A PERCOLATION APPROACH

We fix the parameter p at some value $p \leqslant 0.5$ and go one by one through the sites of an $L \times L$ lattice. For each site, we go through its four neighbors and choose randomly from among their 16 possible configurations a fraction p for which we assign the rule $\sigma_{ij}^{t+1} = 1$. This is done independently for all the sites in the lattice at the beginning of the simulation. These rules, peculiar to each site, are stored for as long as the p parameter value remains the same. That is, at each value of p we are storing a $L \times L \times 16$ matrix. Having fixed the dynamical rules, we now study two random spin configurations $\{\sigma_i\}$ and $\{\rho_i\}$ on a cell of size $L \times L$ which are initially identical except for one edge of the cell, on which the configurations differ for every site on this edge. We have $M_{\text{act}}(t=0) = M_{\text{tot}}(t=0) = L$. We then let the two systems evolve according to the dynamical rules just established for the parameter p, up to some maximum time T. Using the Hoshen–Kopelman method,[9] we now examine the clusters formed on lattice G'' for the existence of a spanning cluster in one direction across the cell. For the same value of p other configurations of $\{\sigma\}$ and $\{\rho\}$ are generated and allowed to evolve according to the same rules up to the same time T and examined for spanning clusters on the lattice G''. The fractions of configurations of $\{\sigma\}$ and $\{\rho\}$ forming spanning clusters on G'' are recorded for the value of p. Another value of p is then chosen and a corresponding set of rules σ_{ij}^{t+1} is established for each of the sites (i, j) and stored. Again the fractions of configurations of $\{\sigma\}$ and $\{\rho\}$ with spanning clusters are determined after evolution up to the same time T.

The threshold value $p_c(L, T)$ is taken to be the value of p for which the fraction of spanning clusters is 1/2. It naturally depends on the system size and maximum evolution time T. Fortunately, we find that for T greater than some value $T_m(L)$ depending on L, the values $p_c(L, T)$ become independent of T up to our numerical accuracy, so that it is not necessary to extrapolate to the infinite-T limit. We find $T_m(L) \approx 10L$. In Table I we present the values of $p_c(L)$ using $T = 10L$. Using finite-size scaling, we can do a least square fit of the values in Table I with the function

$$p_c(L) = p_c(\infty) + aL^{-1/\nu} \tag{2}$$

Table I. Values of the Threshold $p_c(L)$ for the Kauffman Model without Noise Obtained with Maximum Evolution Time $T = 10L$

L	30	40	50	60	70	90	200
$p_c(L)$	0.268	0.276	0.277	0.280	0.282	0.284	0.291

to determine the three parameters $p_c(\infty)$, a, and the correlation length exponent ν. We find $p_c(\infty) = 0.298$ and $\nu^{-1} = 0.6832$. Using this value of ν^{-1}, in Fig. 1 we plot $p_c(L)$ versus $L^{-1/\nu}$. The result fits rather well a straight line with intercept at $p_c(\infty) \approx 0.298$. This confirms finite-size scaling. Our value for $p_c(\infty)$ is consistent with value 0.29 ± 0.01 determined by Stauffer,[6] although it is not entirely clear if this value for p_c must agree with the estimate obtained by requiring the damage to spread through the lattice, without necessarily forming a connected network. Using the value $d_{\text{act}} = d_{\text{tot}} = 1.5$[6] for the fractal dimension of the actual or total damage and the relation $d_{\text{tot}} = 2 - \beta/\nu$, we find $\beta = 0.735$ for the order parameter exponent.

3. KAUFFMAN MODEL WITH NOISE

Inclusion of noise in cellular automata is important since noise plays the role of temperature in equilibrium systems. Thus, a phase transition is expected as the noise level changes for a system with a dimension higher than one. For the deterministic Kauffman model we know that for $p > p_c \approx 0.298$, any initial damage will eventually spread throughout the whole system. The Kauffman model is said to be in the chaotic phase, contrary to the frozen phase, when $p < p_c$ and any finite initial damage remains confined. Here we study the possibility of suppressing this chaotic behavior by introducing noise into the model. We first do this for the most

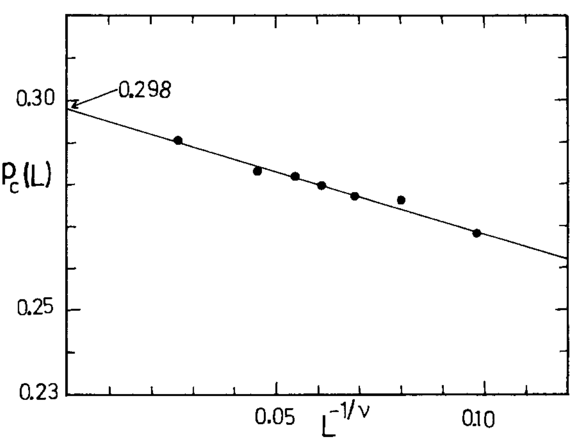

Fig. 1. Values of the threshold $p_c(L)$ for the Kauffman model without noise obtained using maximum evolution time $T = 10L$, plotted versus $L^{-1/\nu}$, with $\nu^{-1} = 0.68$.

chaotic case, $p = 0.5$. Noise is introduced into the model by modifying the dynamical rule (1) to

$$\sigma_{ij}^{t+1} = \begin{cases} f_{ij}(\sigma_{i,j-1}^t, \sigma_{i,j+1}^t, \sigma_{i-1,j}^t, \sigma_{i+1,j}^t; p), \\ \quad \text{with probability } (1-g) \\ \sigma_{ij}^t, \qquad \text{with probability } g \end{cases} \tag{3}$$

with a noise parameter $g \leqslant 1$. For $g = 0$ we recover the deterministic model. With rule (3) we see that for $g = 1$, the initial finite damage will always remain the same. The question is whether there exists some value of g finite but less than 1 at which the system behavior changes from chaotic to frozen. We study the spreading of damage by comparing the time development of two initial spin configurations $\{\sigma\}$ and $\{\rho\}$ on an $L \times L$ square lattice differing at one edge and subject to noise level g, with the parameter p fixed at 0.5. It should be emphasized here that both $\{\sigma\}$ and $\{\rho\}$ are subject to the same sequence of noise, i.e., the same set of random numbers is used in (3) to determine the probability for both $\{\sigma\}$ and $\{\rho\}$ at the same value of g.

Kaneko and Akutsu[10] study the effect of noise on cellular automata using a different rule than (3):

$$\sigma_{i,t}^{t+1} = \begin{cases} f_{ij}(\sigma_{i,j-1}^t, \sigma_{i,j+1}^t, \sigma_{i-1,j}^t, \sigma_{i+1,j}^t; p), \\ \quad \text{with probability } (1-g) \\ 1 - f_{ij}, \qquad \text{with probability } g \end{cases} \tag{4}$$

Since both $\{\sigma\}$ and $\{\rho\}$ are subject to the same noise sequence, it seems that the above rule would not affect the damage. This has in fact been confirmed by a simulation on a $L = 60$ lattice with maximum evolution time $T = 300$. For all noise levels $g \leqslant 1$, the result remains chaotic for all $p \leqslant p_c$. So it seems that for $\{\sigma\}$ and $\{\rho\}$ subjected to the same sequence of noise, (3) is the only way to introduce noise.

For a fixed size L we now let $\{\sigma\}$ and $\{\rho\}$ evolve up to some time T under some noise level g and check if there is a spanning cluster in G''. With everything fixed we now generate different configurations of $\{\sigma\}$ and $\{\rho\}$ and again check for spanning clusters in G''. All these configurations are allowed to evolve up to the same time T and the fraction having spanning clusters is recorded. The critical value $g_c(L, T)$ is determined as the value of the parameter g at which the fraction of configurations with spanning clusters is 1/2 and this obviously depends on the lattice size L and the maximum allowed evolution time T. Contrary to the case of the Kauffman model without noise, we do not find here a maximum time

Table II. Values of the Threshold Noise Level $g_c(L, T)$ for the Kauffman Model with Noise Obtained Using Maximum Evolution Time T, at $p=0.5$

T	$L=60$	$L=100$	$L=150$	$L=200$	$L=\infty$
$5L$	0.845	0.8425	0.833	0.8355	
$10L$	0.920	0.9195	0.9184	0.9180	
$20L$	0.965	0.9645	0.9590	0.9555	
$40L$	0.980	0.9795	0.9793	0.9795	
$80L$	0.990	0.9899	0.9897	0.9855	
$160L$	0.995	0.9947	0.99468	0.9925	
∞	0.99995	0.99945	0.99909	0.9959	0.992

$T_m(L)$ above which $g_c(L, T)$ becomes independent of time T, even though we have gone up to $T=160L$ (i.e., for a 200×200 lattice, we have gone up to 32,000 time steps for every site in the lattice). We give in Table II the value of $g_c(L, T)$ for $L=60$, 100, 150, and 200.

Then for each L we extrapolate the $g_c(L, T)$ values to the $T=\infty$ limit by a least square fit using the function

$$g_c(L, T) = g_c(L, \infty) + bT^{-\phi} \tag{5}$$

We find in all cases $\phi \approx 1.01$. The extrapolated values $g_c(L, \infty)$ are also shown in Table II. We now extrapolate the $g_c(L, \infty)$ values to the $L=\infty$ limit by least square fitting them to the form

$$g_c(L, \infty) = g_c(\infty, \infty) + cL^{-\varphi} \tag{6}$$

with $\varphi = 1/v_g$. We find $g_c(\infty, \infty) \approx 0.992$ and the correlation length exponent v_g to be $v_g^{-1} = 0.41$. We see then that one needs an extremely large noise level in order to force the Kauffman model at $p=0.5$ from the chaotic phase into the frozen phase, even for an infinitely large system. Since our value for $g_c(\infty, \infty)$ for $p=0.5$ is so close to unity, we repeat a simulation at $p=0.35$ and $L=60$ to check if g_c is smaller for the smaller value of p. The value of $g_c(60, T)$ is shown in Table III. These values

Table III. Values of the Threshold Noise Level $g_c(60, T)$ for the Kauffman Model with Noise Obtained Using Maximum Evolution Time T, at $p=0.35$, for a 60×60 Lattice

T	300	600	1200	2400	4800	9600
$g_c(60, T)$	0.77	0.88	0.94	0.97	0.984	0.992

extrapolated to $T=\infty$ give $g_c(60, \infty)$ extremely close to unity. It seems most likely that the result at $p=0.35$ is very similar to that at $p=0.5$. We therefore conclude that for $p \geqslant p_c$ an extremely large noise level would be required to suppress the chaotic behavior, if this is possible at all, even in the limit of an infinitely large system.

Our results has been obtained for the square lattice. It should be interesting to study other types of lattice and also higher dimensionalities. In view of the result obtained here, it is reasonable to believe $g_c=1$ for two dimensions. This is then reminiscent of the situation of the $O(n)$ spin model in two dimensions.

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